

Unitarity, BRST Symmetry and Ward Identities in Orbifold Gauge Theories

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Abstract

We discuss the use of BRST symmetry and the resulting Ward identities as consistency checks for orbifold gauge theories in an arbitrary number of dimensions. We demonstrate that both the usual orbifold symmetry breaking and the recently proposed Higgsless symmetry breaking are consistent with the nilpotency of the BRST transformation. The corresponding Ward identities for 4-point functions of the theory engender relations among the coupling constants that are equivalent to the sum rules from tree level unitarity. We present the complete set of these sum rules also for inelastic scattering and discuss applications to 6-dimensional models and to incomplete matter multiplets on orbifold fixed points.

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1 Introduction

Field theories on a higher dimensional space-time offer new possibilities for symmetry breaking by orbifold Boundary Conditions (BCs) (cf. e.g. [1, 2, 3, 4, 5]) that have been used to construct novel unified theories [6, 7, 8, 9, 10], avoiding common problems of 4-dimensional Grand Unified Theories (GUTs). More general BCs have been used recently in models of Higgsless Electro-Weak Symmetry Breaking (EWSB) [11, 12, 13].

Viewed as effective 4-dimensional theories, compactified higher dimensional field theories are only consistent up to a cutoff usually associated with the scale of a more fundamental theory, e.g. string theory. To make sense of such theories independently of the yet unknown underlying fundamental theory, a minimal requirement is the passing of consistency checks like tree level unitarity below the cutoff and Ward Identities (WIs). Tree level unitarity is to be understood as the requirement that the tree level matrix elements for N -particle scattering amplitudes at high energies scale at most as E^{4-N} . The more restrictive criterion of partial wave unitarity shows that $(4 + N)$ -dimensional gauge theories are valid as effective theories below the scale of $\sim g_D^{-2/N}$ [14, 15], where g_D is the dimensionfull D -dimensional gauge coupling.

As long as a finite number of fields is involved, tree level unitarity of theories involving massive gauge bosons requires Spontaneous Symmetry Breaking (SSB) by the Higgs mechanism [16, 17]. This can be described by a scalar field with coupling constants satisfying appropriate Higgs sum rules [18]. In compactified higher dimensional theories, the Kaluza-Klein (KK) modes of the gauge bosons acquire masses by a geometric Higgs mechanism where the role of the GBs is played by higher dimensional components of the gauge bosons. In 5-dimensional gauge theories, tree level unitarity results from interlacing cancellations within the infinite gauge boson KK-tower [11, 14, 15, 19]. These cancellations rely on relations among the coupling constants that will be called ‘KK-sum rules’ and are helpful in constructing effective 4-dimensional models for massive gauge bosons without Higgs bosons [11]. However, the derivation of these unitarity Sum Rules (SRs) for higher dimensional gauge theories from the explicit calculation of the divergences of the amplitudes is tedious and examples of SRs have so far only been verified in specific models [4, 19] or have been used implicitly in the calculation of partial wave unitarity bounds [14, 15]. A first step toward general SRs has been taken in [11] where two simple KK-SRs for elastic gauge boson scattering have been derived and used to check the consistency of Higgsless symmetry breaking. In 6-dimensional gauge theories that have become popular for SO(10) GUT models [9, 10] and for models of gauge-Higgs unification [20],

the computation of unitarity conditions is further complicated by the apparition of physical scalar components of the gauge fields. The unitarity of such models from the KK point of view has not been discussed so far.

Another example for the importance of unitarity as a consistency check is given by incomplete matter multiplets at the orbifold fixed points. These are one of the key features of orbifold-GUTs that allow to implement a natural suppression of Proton decay and to avoid the doublet-triplet splitting problem. It has been checked for the example of boundary Higgs bosons in a $SU(5)$ theory that this explicit symmetry breaking doesn't cause unitarity violations [4], but a more general discussion of the consistency of this setup has not been given.

However, tree level unitarity by itself is not sufficient for the consistency of the theory. In theories of vector bosons, invariance of the gauge fixed action under a nilpotent BRST transformation is crucial for a consistent quantization with a unitary S -matrix [21]. Therefore the nilpotency of the BRST transformation and checks of BRST invariance of the scattering amplitudes—i. e. checks of the appropriate WIs—are important criteria for the consistency of a theory besides the verification of tree level unitarity.

According to [16, 17], gauge invariance and tree level unitarity are equivalent in Spontaneously Broken Gauge Theories (SBGTs) with renormalizable couplings. To use the WIs as a tool for consistency checks, one should, however, determine a minimal set of amplitudes that has to be checked to ensure the consistency of the theory on tree level. In this paper, we determine a set of WIs that allows for simple, model independent and comprehensive gauge checks [22].

This result allows us to give a much simpler derivation of the conditions following from the unitarity SRs. The WIs are easier to implement both in model building and in checking numerical amplitudes for phenomenological calculations.

In section 2 we introduce the KK-decomposition of the gauge boson lagrangian in an arbitrary number of dimensions and perform the gauge fixing. We verify that both the usual orbifold BCs and the Dirichlet BCs employed in Higgsless EWSB are consistent with the nilpotency of the BRST transformation and allow to define physical states and derive WIs similar to a 4-dimensional SBGT. In section 3 we discuss the use of tree level unitarity and WIs as consistency checks. We review the SRs derived from unitarity and demonstrate that they can be equivalently obtained by imposing simple WIs on the 4-point scattering matrix elements. In section 4 we apply the SRs to unitarity cancellations in 6-dimensional gauge theories, demonstrating the important role of the physical scalar components of the gauge bosons which are a new feature compared to 5-dimensional gauge theories. In section 5 we

apply our results to analyze the consistency of incomplete matter multiplets on orbifold fixed points.

2 BRST-symmetry and Ward Identities in orbifold gauge theories

To derive WIs for a KK-gauge theory that are similar to those of a 4-dimensional SBGTs, one has to choose an appropriate gauge fixing and introduce a nilpotent BRST transformation that leaves the gauge fixed KK-lagrangian invariant. In KK-theories on orbifolds, care has to be taken to use consistent BCs at the orbifold fixed points. In previous discussions [3, 11], consistency with the equations of motion and unitarity has been used as a criterion. Here we will introduce the nilpotency of the BRST transformation as a practical criterion that is much easier to use than the calculation of unitarity violating terms of scattering amplitudes in [11].

2.1 Kaluza Klein decomposition of a Z_2^n orbifold gauge theory

To establish our notation and to introduce an appropriate gauge fixing, we perform the KK-decomposition of a Z_2 orbifold gauge theory in $4 + N$ dimensions. Subsequently we will derive the WIs and determine BCs consistent with the nilpotency of the BRST transformation.

For definiteness, we assume a factorisable constant metric of the $(4 + N)$ -dimensional space-time of the form $g_{AB} = \text{diag}(\eta_{\mu\nu}, -\gamma_{ij})$. The generalization to a warped background metric by plugging appropriate warp factors into the formulae below is straightforward. Our notation for the Yang-Mills lagrangian in $4 + N$ dimensions is given in appendix A.

The KK-decomposition of the gauge fields is

$$A_A^a(x, y) = \begin{pmatrix} A_\mu^a(x, y) \\ \Phi_i^a(x, y) \end{pmatrix} = \sum_{\vec{n}} \begin{pmatrix} f_{\vec{n}}^a(y) A_{\vec{n}, \mu}^a(x) \\ g_{\vec{n}}^a(y) \Phi_{\vec{n}, i}^a(x) \end{pmatrix} \quad (1)$$

where the wavefunctions $\chi = f, g$ satisfy the differential equation

$$\partial_i \partial^i \chi_{\vec{n}}^a(y) = -m_{\vec{n}}^{a2} \chi_{\vec{n}}^a \quad (2)$$

They are chosen orthonormal and satisfying a completeness relation:

$$\int d^N y \chi_{\vec{n}}^a(y) \chi_{\vec{m}}^a(y) = \delta_{\vec{n}, \vec{m}}, \quad \sum_{\vec{n}} \chi_{\vec{n}}^a(x) \chi_{\vec{n}}^a(y) = \delta(y - x) \quad (3)$$

(the group indices a are not summed over here). To determine the BCs imposed on the wavefunctions, let us recall some aspects of symmetry breaking on orbifolds [2, 3, 4, 5]. An orbifold C/K is obtained from a compact manifold C and a discrete group K by identifying points $y \in C$ under the action of K , i.e. $y \simeq P_k y$, where the P_k form a representation of K on C . In an orbifold the action of K has a set of fixed points $\{y_f\}$

$$y_f = P_k y_f \quad (4)$$

for some $k \in K$. (For more than one extra dimension, there can also be fixed lines, fixed surfaces etc.). Fields defined on an orbifold need only be invariant under K up to transformations Z_k of a symmetry group of the lagrangian that form a representation of K in field space:

$$\Phi(P_k y) = Z_k \Phi(y) \quad (5)$$

For definiteness, we will discuss the most familiar example of an orbifold symmetry: the group $K = Z_2$ that is generated by one element $P(y - y_f) = -(y - y_f)$. In this case¹ the transformations of the gauge fields take the form

$$\begin{aligned} A_\mu^a(x, Py) &= \eta^a A_\mu^a(x, y) \\ \Phi_i^a(x, Py) &= -\eta^a \Phi_i^a(x, y) \end{aligned} \quad (6)$$

where the η^a are the eigenvalues of the representation matrix Z . The transformation law of the higher dimensional components is determined from the requirement of a homogeneous transformation of the covariant derivative. For $\eta^a \neq 1$, the 4-dimensional gauge fields must vanish at the fixed points y_f , i.e. they must satisfy Dirichlet BCs. Therefore the gauge symmetry is broken to a subgroup H_k at the boundary. The different parity of the scalar components implies that they satisfy Neumann BCs. For $\eta^a = 1$ the symmetry remains unbroken and the BCs of 4-dimensional vectors and scalars are exchanged. This can be summarized as (identifying the indices corresponding to broken generators by a hat):

$$\begin{aligned} A_\mu^{\hat{a}}(x, y_f) &= 0 & \partial_i \Phi_j^{\hat{a}}(x, y_f) &= 0 \\ \partial_i A_\mu^a(x, y_f) &= 0 & \Phi_i^a(x, y_f) &= 0 \end{aligned} \quad (7)$$

¹For more complicated orbifold symmetries like T^2/Z_4 in 6 dimensions [20], the orbifold transformation can mix higher dimensional components, since a homogeneous transformation of the covariant derivative requires the transformation law $\Phi_i^a(x, P_k y) = \eta_k^a \Phi_j^a(x, y) (P_k^{-1})_{ji}$. Therefore the BCs of the Φ_i depend on the matrix P_k and the general description of the KK-decomposition becomes more involved.

Only gauge fields that remain unbroken at every fixed point have KK-zero-modes with vanishing masses. If desired, zero-modes of the higher dimensional scalar components of the broken gauge fields that survive the orbifolding (6) can be projected out by introducing further orbifold symmetries.

The conditions (7) translate into the boundary conditions for the KK-wavefunctions

$$\begin{aligned} f^{\hat{a}}(y_f) &= 0 & \partial_i g^{\hat{a}}(y_f) &= 0 \\ \partial_i f^a(y_f) &= 0 & g^a(y_f) &= 0 \end{aligned} \quad (8)$$

Because they satisfy the same BCs, the derivatives $\partial_i f$ can be expanded in the basis of the g and the $\partial_i g$ can be expanded in the basis of the f . One can choose the g such that

$$\partial_i f_{\vec{n}} = m_{n_i} g_{\vec{n}} \quad , \quad \partial_i g_{\vec{n}} = -m_{n_i} f_{\vec{n}} \quad (9)$$

with $\sum_i m_{n_i}^2 = m_{\vec{n}}^2$. This is consistent with the equation of motion (2) and will diagonalize the gauge boson masses and coupling between the A_μ and the Φ_i . For the familiar torus compactification on orbifolds, the property (9) is obviously satisfied and a similar relation has been imposed for the warped case in [23].

The effective 4-dimensional lagrangian can now be derived using the KK-decomposition (1) and exploiting the relations (9). To avoid notational clutter, we introduce multi-indices $\alpha \equiv (a, \vec{n})$ and $\alpha_i \equiv (a, 0, \dots, 0, n_i, 0, \dots, 0)$ and use a summation convention also for the sum over the KK-states. Using the relations (9), we find the cubic interaction terms

$$\begin{aligned} \mathcal{L}_{\text{cubic}}^{KK} &= -g^{\alpha\beta\gamma} \partial_\mu A_\nu^\alpha A^{\beta,\mu} A^{\gamma,\nu} - \frac{1}{2} T_{\beta\gamma}^\alpha A^{\alpha,\mu} \Phi^{\beta,i} \overleftrightarrow{\partial}_\mu \Phi_i^\gamma \\ &\quad + \frac{1}{2} g_{\Phi AA}^{i\alpha\beta\gamma} \Phi^{\alpha,i} A_\mu^\beta A^{\mu,\gamma} - \frac{1}{2} T_{\beta\gamma}^\alpha (m_{\alpha_j} \Phi_i^\alpha - m_{\alpha_i} \Phi_j^\alpha) \Phi^{\beta,i} \Phi^{\gamma,j} . \end{aligned} \quad (10)$$

The remaining terms in the KK-lagrangian are given in (82) in the appendix. The coupling constants are products of group theory factors and integrals over products of KK-wavefunctions, e.g.:

$$g^{\alpha\beta\gamma} = f^{abc} \int d^N y f^\alpha(y) f^\beta(y) f^\gamma(y) \quad (11a)$$

$$T_{\beta\gamma}^\alpha = f^{abc} \int d^N y f^\alpha(y) g^\beta(y) g^\gamma(y) . \quad (11b)$$

The explicit form of the other couplings is given in (83). Note that the interactions among the scalars originate from the F_{ij} components and appear only in more than one extra dimension.

The KK-decomposition yields a bilinear mixing $-m_{\alpha_i}\partial_\mu\Phi^{\alpha,i}A^{\alpha,\mu}$ of 4-dimensional gauge bosons and scalars. Therefore at each KK-level the linear combination

$$\phi^\alpha \equiv -\frac{m_{\alpha_i}}{m_\alpha}\Phi^{\alpha,i} \quad (12)$$

plays the role of a geometric Goldstone boson that is eaten by the KK gauge bosons, leaving $N - 1$ physical scalars. The sign in (12) is chosen because of compatibility with our conventions for the WIs in 4-dimensional theories.

Accordingly, we decompose the scalars into the GBs (12) and ‘geometric Higgs’ bosons:

$$\Phi_i^\alpha = H_i^\alpha - \frac{m_{\alpha_i}}{m_\alpha}\phi^\alpha \quad (13)$$

where H^i is defined as orthogonal to the GBs, i.e. $m_{\alpha_i}H^{\alpha,i} = 0$. The mass term for the scalar obtained from the KK-decomposition of the F_{ij}^2 term of the lagrangian has the form

$$m_\alpha^2\Phi^{\alpha,i^2} - (m_{\alpha_i}\Phi^{\alpha,i})^2 = m_\alpha^2H^{\alpha,i^2} \quad (14)$$

and the GBs are massless, as they must be. To eliminate the mixing of gauge bosons and GBs, we choose a gauge fixing function

$$G^a = -\frac{1}{\xi}(\partial_\mu A^{a,\mu}(x, y) - \xi\partial_i\Phi^{a,i}(x, y)) \quad (15)$$

that extends the one introduced in [24, 14] to more than one extra dimension. In terms of KK-modes, the gauge fixing lagrangian takes the form

$$\mathcal{L}_{GF} = -\int d^N y \frac{1}{2\xi} G^{a2} = -\frac{1}{2\xi}(\partial_\mu A^{\alpha,\mu} - \xi m_\alpha \phi^\alpha)^2 \quad (16)$$

2.2 BRST-Symmetry and consistent boundary conditions

The possible symmetry breaking patterns resulting from orbifold BCs (6) are highly constrained [3]. For example, neither EWSB nor the breaking of $SO(10)$ to the SM is possible in 5 dimensions by abelian orbifold conditions alone. These constraints arise from the relation

$$f^{abc} = \eta_k^a \eta_k^b \eta_k^c f^{abc} \quad (17)$$

following from the requirement that the field strength transforms according to $F_{\mu\nu}^a \rightarrow \eta_k^a F_{\mu\nu}^a$. Because of (17), only structure constants with an even number of broken indices can be nonvanishing.

In order to liberate model builders from these constraints, mixed BCs of the form

$$\partial_i A_\mu^a(y_f) = V_{y_f}^{ab} A_\mu^b(y_f) \quad (18)$$

have been proposed in [3]. They can be introduced consistently by coupling the gauge fields to a Higgs boson at the boundary [3, 11]. However, imposing such BCs without including the Higgs boson engenders unitarity violations. Taking the vacuum expectation value of the boundary Higgs to infinity, the mixed BCs turn into Dirichlet BCs

$$A_\mu^b(y_f) = 0 \quad (19)$$

that maintain the unitarity of gauge boson scattering, while avoiding the constraints from orbifold symmetry breaking. This possibility has been utilized for Higgsless EWSB by BCs alone [11, 12, 13].

To investigate the mixed and Dirichlet BCs further, we will use the nilpotency of the BRST quantization as a consistency check. To derive the BRST transformations of the KK-modes we use that consistency of the $(4 + N)$ -dimensional BRST transformation of the gauge boson

$$\delta_{\text{BRST}} A_M^a(x, y) = \partial_M c^a(x, y) + f^{abc} A_M^b(x, y) c^c(x, y) \quad (20)$$

demands that ghosts must satisfy the same BCs as the 4-dimensional components of the gauge bosons and therefore have a KK-decomposition in terms of the f^α . From the BRST transformation of the antighost together with the equation of motion of the auxiliary field, it can be inferred that the same wavefunctions appear also in the KK decomposition of the antighosts. The complete set of BRST transformations in $(4 + N)$ -dimensions is given in (80). The BRST transformations of the KK-modes are then given by

$$\delta_{\text{BRST}} A_\mu^\alpha(x) = \partial_\mu c^\alpha(x) + g^{\alpha\beta\gamma} A_\mu^\beta(x) c^\gamma(x) \quad (21a)$$

$$\delta_{\text{BRST}} \Phi_i^\alpha(x) = m_{\alpha_i} c^\alpha(x) + T_{\alpha\beta}^\gamma \Phi_i^\beta(x) c^\gamma(x) \quad (21b)$$

$$\delta_{\text{BRST}} c^\alpha(x) = -\frac{1}{2} g^{\alpha\beta\gamma} c^\beta(x) c^\gamma(x) \quad (21c)$$

$$\delta_{\text{BRST}} \bar{c}^\alpha(x) = B^\alpha(x) \quad (21d)$$

$$\delta_{\text{BRST}} B^\alpha(x) = 0 \quad (21e)$$

and the gauge fixing (15) implies the equations of motion of the KK-modes of the auxiliary field B :

$$B^\alpha = -\frac{1}{\xi} G^\alpha = -\frac{1}{\xi} (\partial_\mu A^{\alpha,\mu} - \xi m_\alpha \phi^\alpha) \quad (22)$$

The appearance of the inhomogeneous term in the BRST transformations (21b) of the higher dimensional components of the gauge fields supports the interpretation of (12) as GBs. The remaining transformation laws agree with those of a gauge theory with (an infinite number of) ‘structure constants’ $g^{\alpha\beta\gamma}$ and ‘generators’ $T_{\alpha\beta}^\gamma$.

The BRST transformations are nilpotent iff the relations

$$T_{\alpha\beta}^\gamma T_{\beta\epsilon}^\delta - T_{\alpha\beta}^\delta T_{\beta\epsilon}^\gamma = g^{\beta\delta\gamma} T_{\alpha\epsilon}^\beta \quad (23a)$$

$$m_{\beta_i} T_{\alpha\beta}^\gamma - m_{\gamma_i} T_{\alpha\gamma}^\beta = m_{\alpha_i} g^{\alpha\beta\gamma} \quad (23b)$$

$$g^{\alpha\beta\epsilon} g^{\gamma\delta\epsilon} + g^{\gamma\alpha\epsilon} g^{\beta\delta\epsilon} + g^{\alpha\delta\epsilon} g^{\beta\gamma\epsilon} = 0 \quad (23c)$$

hold. Inserting the definitions (11a) and (11b) and using the relations (9) we find that the condition (23b) (a similar relation occurs in a 4-dimensional SBGT, cf. (37a)) is equivalent to

$$0 = f^{abc} \int d^N y \partial_i (g^\alpha f^\beta f^\gamma) = f^{abc} [g^\alpha f^\beta f^\gamma]_{y_f} \quad (24)$$

For three unbroken indices, the boundary term vanishes, because g^α is zero on the boundary. For one broken index, the structure constants vanish, because the unbroken generators must close into an algebra. For two or three broken indices, there is at least one broken wavefunction $f^{\hat{\alpha}}$. Thus (23b) is satisfied as long as the broken wavefunctions vanish on the boundary. This shows that general Dirichlet BCs (19) are consistent, in contrast to the mixed BCs (18). This confirms the results obtained from unitarity in [11].

The ‘Lie Algebra’ (23a) and the ‘Jacobi Identity’ (23c) are satisfied automatically by the Jacobi identity of the structure constants f^{abc} . This can be seen using the completeness relations of the KK-wavefunctions (cf. also [11]). Inserting the definitions of the coupling constants (11) and suppressing the group indices for the moment, every term of (23a) and (23c) involves an expression of the form

$$\begin{aligned} \sum_{\vec{n}} \int d^N y \chi_{\vec{n}}(y) \chi_{\vec{m}}(y) \chi_{\vec{l}}(y) \int d^N y' \chi_{\vec{n}}(y') \chi_{\vec{k}}(y') \chi_{\vec{p}}(y') \\ = \int d^N y \chi_{\vec{m}}(y) \chi_{\vec{l}}(y) \chi_{\vec{k}}(y) \chi_{\vec{p}}(y) \end{aligned} \quad (25)$$

Here we have used the completeness relations (3) to get rid of the sum over the KK-modes. As an example, the RHS of (23a) can be put in the form

$$g^{\beta\delta\gamma} T_{\alpha\epsilon}^\beta = f^{bdc} f^{bae} \int d^N y f^\delta(y) f^\gamma(y) g^\alpha(y) g^\epsilon(y) \quad (26)$$

Performing similar manipulations for the remaining terms, the same integral over the KK-wavefunctions appears everywhere and (23a) is reduced to the Jacobi Identity for the f^{abc} . The identity (23c) can be treated accordingly.

Having ensured the nilpotency of the BRST-charge, we can perform the gauge fixing as usual by adding the BRST transform of a functional of ghost number -1 :

$$\mathcal{L}_{GF} + \mathcal{L}_{FP} = \delta_{\text{BRST}} \left[\bar{c}_\alpha (G^\alpha + \frac{\xi}{2} B^\alpha) \right] \quad (27)$$

with the gauge fixing functional G^α as defined in (15). Having a BRST invariant gauge fixed lagrangian and a nilpotent BRST charge, the physical states can be defined as usual by the Kugo-Ojima condition

$$Q |\psi_{\text{phys}}\rangle = 0 \quad (28)$$

Using the equation of motion for the auxiliary field B^a (22), we obtain the WI

$$0 = \langle \phi_{\text{phys}} | \{Q, \bar{c}^\alpha\} | \psi_{\text{phys}} \rangle = -\frac{1}{\xi} \langle \phi_{\text{phys}} | (\partial_\mu A^{\alpha,\mu} - \xi m_\alpha \phi^\alpha) | \psi_{\text{phys}} \rangle \quad (29)$$

To turn this into an identity for scattering matrix elements, one needs to amputate the external gauge boson and GB propagator. With our choice of gauge fixing, the usual R_ξ gauge tree level relation of the propagators [25]

$$k_\mu D_A^{\mu\nu} = -\xi D_\phi k^\nu \quad (30)$$

is satisfied² and the amputation results in the WI for scattering matrix elements

$$-i p_{a\mu} \mathcal{M}^\mu(A^\alpha(p_a) \dots) - m_\alpha \mathcal{M}(\phi^\alpha(p_a) \dots) \equiv \mathcal{M}(\mathcal{D}_a(p_a) \dots) = 0 \quad (31)$$

that is similar to the WI in a 4D SGBT. This will allow us in section 3 to use the SRs obtained from the WIs in 4-dimensional theories also for KK-theories.

3 Unitarity Sum Rules and Ward Identities

The WIs (31) implied by BRST symmetry provide a powerful tool for consistency checks in gauge theories. An advantage over tree unitarity as consistency criterion is the simpler applicability due to the fact that the WIs

²In higher orders, loop corrections to (30) have to be considered similar to the 4-dimensional case [26].

hold at every point in phase space, independent of the external momenta. In contrast, tree level unitarity requires to take the high energy limit of the momenta. Therefore WIs are also more powerful as consistency checks in numerical calculations [22].

As we demonstrate in this section, the SRs of the coupling constants implied by tree level unitarity, along with additional relations for GB couplings, can also be obtained from a suitable set of WIs for 4-point functions. Our discussion is valid for general field theories with the field content of a SBGT and dimension four couplings, including orbifold gauge theories. We establish our result by computing the WIs using a general parametrisation of a lagrangian for such a theory without assuming the symmetry relations resulting from a spontaneously broken gauge symmetry. A comparison with the unitarity SRs [16, 17, 18] then shows that both approaches yield the same results. The applicability of the SRs to KK-gauge theories is discussed in subsection 3.3.

The key formula derived below and used in the discussion of unitarity in 6-dimensional gauge theories in section 4 is the SR (39). The discussion of incomplete matter multiplets on the boundary in section 5 makes use of the Lie algebra structure of the matter gauge couplings (35a). Apart from serving as consistency checks, the SRs discussed in this section provide also tools in model building. In the context of non minimal Higgs models, this has been discussed already in [18] while for models of EWSB without a Higgs, a simple SR obtained from elastic scattering has been used in [11]. As a possible future application of the SRs, it would be interesting to use the SRs to explore 4-dimensional UV completions of the Higgsless models of [11, 12], for instance by truncating the KK tower and introducing heavy scalars with appropriate couplings determined by the SRs to cancel unitarity violations in the scattering of the higher KK-levels.

3.1 A minimal set of Ward Identities

To use the WIs (31) of SBGTs as a tool for consistency checks, one should determine a minimal set of amplitudes that has to be checked to ensure the consistency of the theory. Since the Feynman rules of SBGTs are determined by tree level unitarity of 4 particle scattering amplitudes [17] (apart from the scalar self interaction that make it necessary to consider some 5-point amplitudes), it is reasonable to expect that a similar set of amplitudes is sufficient for the consistency checks using the WIs. However, this connection has never been made precise in the literature.

In an unbroken gauge theory, it is not difficult to see that the Yang-Mills structure of the lagrangian can indeed be ‘reconstructed’ by imposing the WI

on the 4-point amplitudes. It is a well known textbook example, that the WI of the two quark-two gluon amplitude implies the Lie algebra relations of the quark-gluon couplings. Similarly one can show that the WI of the 4-gluon amplitude determines the quartic gluon coupling and implies the Jacobi Identity for the triple gluon coupling. As discussed in subsection 3.2, the same relations are also obtained in SBGTs.

However, in SBGTs in R_ξ gauge, the determination of the *complete* set of Feynman rules from WIs with one contraction is more involved, since amplitudes with external GBs have to be considered. Because of the unphysical nature of the GBs, such WIs include ghost terms in addition to (31) and further consistency checks would be required to determine the ghost Feynman rules. As we will show below, such complications can be avoided by considering the generalized WIs

$$\mathcal{M}(\mathcal{D} \dots \mathcal{D}\Phi \dots \Phi) = 0 \quad (32)$$

with up to 4 contractions that allow to reconstruct the Feynman rules of SBGTs (apart from quartic Higgs selfcouplings) from the *limited* set of WIs for the 4-point functions without external GBs. This allows for simple, model independent and comprehensive gauge checks that avoid the introduction of ghosts already on tree level.

Since the lagrangian of a KK-gauge theory has the same form as that of a SBGT and the same WIs (31) hold, we can then infer that the same SRs are also valid for a KK-theory. However, the need to truncate potentially divergent sums over KK-modes might spoil the unitarity cancellations, this is discussed in subsection 3.3.

3.2 Unitarity sum rules from Ward Identities

To present the results of the computation of the WIs, we have to introduce a parametrization of the general renormalizable Lagrangian with the particle spectrum of a SBGT but *without* imposing gauge invariance. We denote gauge bosons by W^a , Goldstone bosons by ϕ_a and all other scalar fields by H_i . We will not introduce a separate notation for massless gauge bosons and take it as understood that the GBs are only associated to massive gauge bosons. The couplings that will be most relevant below are defined as

$$\begin{aligned} \mathcal{L}_{\text{int}} = & -f^{abc}W_{b,\mu}W_{c,\nu}\partial^\mu W_a^\nu + \frac{1}{2}g_{\phi WW}^{Aab}\phi_A W_{a,\mu}W_b^\mu - \frac{1}{2}T_{AB}^a(\phi_A \overleftrightarrow{\partial}_\mu \phi_B)W_a^\mu \\ & + \bar{\psi}_i W_a \left[\tau_{Lij}^a \left(\frac{1-\gamma^5}{2} \right) + \tau_{Rij}^a \left(\frac{1+\gamma^5}{2} \right) \right] \psi_j + \bar{\psi}_i \phi_A \left[X_{ij}^A \left(\frac{1-\gamma^5}{2} \right) + X_{ij}^{A\dagger} \left(\frac{1+\gamma^5}{2} \right) \right] \psi_j \end{aligned} \quad (33)$$

Here we have collected GBs and physical scalars in a vector $\phi_A = (\phi_a, H_i)$. The components of the coupling matrices of the scalars will be denoted as

$$\mathbf{T}^c = \begin{pmatrix} t^c & -g_{H\phi W}^c \\ (g_{H\phi W}^c)^T & T^c \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} g_\phi^a \\ g_H^k \end{pmatrix} \quad (34)$$

The complete lagrangian in this notation is given in appendix B. The translation to the notation used in the specific example of the KK-theory in (10) should be clear.

We use the lagrangian (33) without further assumptions in the evaluation of the STIs. In order to describe a SBGT, however, the couplings appearing in (33) must satisfy certain invariance conditions. Apart from the familiar Jacobi Identity of the f^{abc} and the Lie algebra of the fermion gauge couplings

$$[\tau_{L/R}^a, \tau_{L/R}^b]_{ij} = i f^{abc} \tau_{L/R}^c \quad (35a)$$

also the gauge couplings of the scalars must satisfy a Lie algebra and the Yukawa couplings must be invariant under global transformations generated by the τ and T :

$$[\mathbf{T}^a, \mathbf{T}^b] = f^{abc} \mathbf{T}^c \quad (35b)$$

$$-i\tau_R^a \mathbf{X} + i\mathbf{X} \tau_L^a = \mathbf{T}^a \mathbf{X} \quad (35c)$$

There are also constraints on the quartic couplings, for instance the quartic couplings of the scalars to gauge boson in a SBGT have the form

$$g_{\phi^2 W^2}^{ABcd} = -\{\mathbf{T}^c, \mathbf{T}^d\}_{AB} \quad (36)$$

Furthermore, the condition that some of the scalars H_i acquire a vacuum expectation value that is responsible for gauge boson and fermion masses allows to express the couplings $g_{\phi WW}$ in terms of the \mathbf{T} and the masses and furthermore to eliminate all the GB couplings as independent parameters (cf. appendix C). With these identifications, the relations (35) and the component $g_{H^2 W^2}$ of (36) are precisely the unitarity SRs derived in [16, 17, 18].

We now demonstrate how the unitarity SRs (35) and (36) can be obtained also from WIs of four point functions. Since this requires the somewhat tedious task to expand the SRs in component form and compare to the results of the WIs, we have collected the explicit expressions in appendix D. Details on the calculation of the WIs can be found in [22].

In a first step, we evaluate the WIs for the 3-point matrix elements and find they determine the form of the cubic couplings involving GBs, reproducing the expressions in appendix C. As examples, the results of these WIs

determine the GB components of the matrices (34)


$$t_{ac}^b = -\frac{1}{2m_{W_a}m_{W_c}}f^{abc}(m_{W_b}^2 - m_{W_a}^2 - m_{W_c}^2) \quad (37a)$$

$$m_{W_a}g_{H\phi W}^{iab} = -\frac{1}{2}g_{HWW}^{iab} \quad (37b)$$

$$m_{W_a}g_{\phi ij}^a = i(m_j\tau_{Rij}^a - m_i\tau_{Lij}^a) \quad (37c)$$

While the same definitions appear in the unitarity approach as auxiliary quantities [16], the physical content of (37) remains, however, rather obscure in this approach because they are not identified with GB couplings. The remaining relations obtained from the 3 point WIs are given in appendix D.1 and include conditions that are a consequence of the gauge invariance of the scalar potential and haven't been derived from unitarity in [16].

We now turn to the evaluation of WIs for 4-point scattering amplitudes with one contraction. The results can be found in appendix D.2 and include consistency relations among the coupling constants of the physical particles like the Lie algebra structure of the fermion couplings (35a), the Higgs component of the Lie algebra (35b), the invariance condition of the Higgs-Yukawa coupling contained in (35c) or the expression for the $WWHH$ coupling in (36). As an example, the WI for the 4 gauge boson amplitude implies the Jacobi Identity of the f^{abc} and determines the quartic gauge boson coupling:



$$\Rightarrow \begin{cases} f^{abe}f^{cde} + f^{cae}f^{bde} + f^{ade}f^{bce} = 0 \\ g_{W^4}^{abcd} = f^{abe}f^{cde} - f^{ade}f^{bce} \end{cases} \quad (38)$$

In the diagrammatical representation of the WIs, the insertion of the operator $(\partial_\mu W^\mu - m_W \phi)$ is represented by a double line.

As a general rule, the results obtained from the WIs with one contraction correspond to the SRs ensuring the cancellation of the leading divergencies in the unitarity approach [16, 17, 18]. The relations (38), for instance arise from the cancellation of the terms growing like E^4 in gauge boson scattering.

Turning to the results of the WIs with more than one contraction, it turns out that they include the remaining components of the Lie algebra (35b) and the Yukawa transformation law (35c). These are the SRs ensuring the cancellation of the subleading divergences in the unitarity approach so indeed all the SRs ensuring tree level unitarity can be obtained in a simpler way from the WIs. As an example, let us give the relation that ensures the cancellation of the subleading divergencies $\propto E^2$ in gauge boson scattering [16, 17, 18] that

we will verify in a 6-dimensional KK-theory in section 4. In our approach, it arises from the WI for the 4-gauge boson amplitude with three contractions and is given by

$$\begin{aligned}
 \text{Diagram} & \Rightarrow \left\{ \frac{1}{m_{W_e}^2} [f^{abe} f^{ced} (m_{W_a}^2 - m_{W_b}^2 - m_{W_e}^2)(m_{W_c}^2 - m_{W_e}^2 - m_{W_d}^2)] \right. \\
 & \left. - g_{HWW}^{iab} g_{HWW}^{icd} \right\} - a \leftrightarrow c = 2f^{ace} f^{ebd} (m_{W_e}^2 - m_{W_b}^2 - m_{W_d}^2)
 \end{aligned} \tag{39}$$

We take it as understood that the internal summations involving inverse gauge boson masses extend only over massive gauge bosons. The resulting equation for elastic scattering has been used in [11] to check the consistency of EWSB by Dirichlet boundary conditions in 5 dimensions. Using the result (37a) for the GB-gauge boson coupling, one can check that (39) is the ab component of the Lie-algebra of the scalar couplings (35b).

Apart from these relations ensuring unitarity of gauge boson scattering, the results of the remaining WIs of 4-point functions given in appendix D include further conditions determining the GB couplings that have not been obtained from unitarity in [16, 17, 18]. At one hand, these are the conditions

$$T_{AB}^a g_{\phi WW}^{Bbc} - m_{W_a} g_{\phi^2 W^2}^{Aabc} = f^{acd} g_{\phi WW}^{Abd} + f^{abd} g_{\phi WW}^{Acd} \tag{40}$$

that fix the quartic GB-gauge boson couplings. This relations follow from the graded Jacobi Identity

$$0 = [A, \{B, C\}] + \{[C, A], B\} - \{[A, B]C\} \tag{41}$$

and the form of the 2-gauge boson 2-scalar interaction (36).

The remaining WIs of 4 point functions imply the condition for the gauge invariance of the scalar potential:

$$m_{W_a} g_{\phi^4}^{ABCa} + g_{\phi^3}^{DBC} T_{DA}^a + g_{\phi^3}^{ADC} T_{DB}^a + g_{\phi^3}^{ABD} T_{DC}^a = 0 \tag{42}$$

This relation determines the quartic couplings of the GBs in terms of already known quantities. Apart from the quartic Higgs selfcouplings $g_{\phi^4}^{ijkl}$ that are not included in (42), the WIs of four point functions therefore determine the relations among the coupling constants of a SBGT and fix the GB couplings in terms of the couplings of the physical particles.

3.3 Sum rules in KK-theories

The KK-decomposition of a higher dimensional gauge theory results in an effective 4-dimensional lagrangian of the same general form (84), albeit with

an infinite number of fields. As discussed in subsection 2.2, the gauge fixing function (15) and a consistent set of BCs ensures the validity of WIs similar to those of a 4-dimensional SBGT and the same SRs also hold in KK-theories. However, in calculations of scattering amplitudes, sums over KK-modes appear that can be infinite if KK-momentum conservation breaks down. For more than one extra dimension, the sums over the propagators diverge and a cutoff has to be introduced. Therefore the connection of the SRs to unitarity requires further clarification for KK-theories.

Let us first note that independently of the convergence properties of the amplitudes, no divergencies appear in the unitarity SRs, even though they involve—like the conditions (23) considered in subsection 2.2—potentially infinite sums over products of ‘generators’ and coupling constants from (11). However, just as in (26) we can use the completeness relations (3) and the sum over the KK-modes drops out of all SRs and no problems with divergences arise. To infer that the SRs obtained from the 4-point WIs also imply tree level unitarity for the scattering amplitudes of a KK-gauge theory, we have to use, however, that the sum over the KK-modes converges, since a regularization procedure can lead to a violation of the SRs. For a sharp cutoff of the sum, the SRs cannot be satisfied for the highest KK-level [11]. The introduction of a smooth regulator in the KK-sum corresponds to a modification of the KK-propagators. Because the gauge (and unitarity) cancellations rely on WIs connecting contracted vertices and inverse propagators, they can be spoiled by such a modification.

We expect however, that this poses no serious problem in realistic applications: As in 4-dimensional SBGTs, the unitarity SRs doesn’t imply partial wave unitarity. Instead of the usual 4-dimensional bound on the Higgs mass ~ 1 TeV [27], in the 5-dimensional SM there is an upper bound of the number of KK-modes $N_{KK} \sim 10$ [14] compatible with unitarity. Since this bound is rather small compared the scale where problems from diverging KK-sums arise, partial wave unitarity is expected to give a more stringent criterion for the breakdown of unitarity than the disturbance of unitarity cancellations by a cutoff.

In general, since problems with unitarity are associated with longitudinally polarized massive gauge bosons, a conflict of unitarity and regularization can arise only in models where the couplings of gauge bosons violate KK-selection rules in such a way that an infinite number of modes contributes to scattering processes with external gauge bosons. Furthermore there must be more than one extra dimension allowing the KK-sums to diverge. The models considered in sections 4 and 5 are not affected by such problems. However, these issues deserve further studies, especially because one expects that possible violations of gauge invariance by a cutoff will play

a more important role in loop calculations.

4 Unitarity Sum Rules in 6 dimensions

As an application of the unitarity SRs, we will discuss gauge boson scattering in a 6-dimensional gauge theory. The new ingredients compared to the 5-dimensional theories discussed in [11, 14, 15, 19] are the physical scalar component of the bulk gauge bosons (cf. (13)) and the apparition of more than one KK-state at the same level. As we will demonstrate, these two phenomena are connected and the physical scalars play an important role in the unitarity cancellations if the initial and final state KK-momenta are not parallel.

While the SRs considered in this work only ensure the correct scaling behavior implied by unitarity, we expect our results also to be useful as a first step in a partial wave analysis along the lines of [14, 15]. Other interesting possible extensions of our work include a compactification with different radii and nontrivial shape parameters [28], the introduction of additional orbifold fixed points for the application to GUT models [9, 10] and more complicated orbifold symmetries like Z_4 [20].

4.1 Kaluza-Klein decomposition and Feynman rules

For simplicity, we consider a 6-dimensional gauge theory compactified on a torus with identical radii without symmetry breaking by orbifold BCs. Before orbifolding, the KK-decomposition of an arbitrary field Φ is given by

$$\Phi(x, y) = \frac{1}{2\pi R} \sum_{\vec{n}} \Phi_{\vec{n}}(x) e^{\frac{i}{R}(\vec{n} \cdot \vec{y})} \quad (43)$$

where $\vec{n} = (n_1, n_2)$. The KK-modes satisfy $\Phi_{\vec{n}}^* = \Phi_{-\vec{n}}$ for hermitian fields.

According to (12), the unphysical GB and the orthogonal physical component are [9]

$$\phi_{\vec{n}}^a = -\frac{1}{Rm_{\vec{n}}} (n_1 \Phi_{\vec{n}}^{a,1} + n_2 \Phi_{\vec{n}}^{a,2}) \quad (44a)$$

$$H_{\vec{n}}^a = \frac{1}{Rm_{\vec{n}}} (-n_2 \Phi_{\vec{n}}^{a,1} + n_1 \Phi_{\vec{n}}^{a,2}) \quad (44b)$$

We will limit our discussion to a simple orbifolding [29] T^2/Z_2 with $Z_2 : (y, z) \rightarrow (-y, -z)$ where all gauge symmetries remain unbroken. This imposes the conditions $\Phi_{n_1, n_2}^+ = \Phi_{-n_1, -n_2}^+$ on the KK-modes with positive Z_2

parity, while the negative parity modes must satisfy $\Phi_{n_1, n_2}^- = -\Phi_{-n_1, -n_2}^-$. The range of the coordinates in the orbifold can be taken as [29] $-\pi R \leq y_1 \leq \pi R$ and $0 \leq y_2 \leq \pi R$.

The decompositions of the vector and scalar fields become:

$$A^{a, \mu}(x, y) = \frac{1}{\sqrt{2\pi R}} A_0^{a, \mu}(x) + \frac{1}{\pi R} \sum_{\vec{n}}^{\infty} \cos\left(\frac{n_i y_i}{R}\right) A_{\vec{n}}^{a, \mu}(x) \quad (45)$$

$$\Phi^{a, i}(x, y) = \frac{1}{\pi R^2} \sum_{\vec{n}} \frac{1}{m_{\vec{n}}} \sin\left(-\frac{n_i y_i}{R}\right) \Phi_{\vec{n}}^{a, i}(x) \quad (46)$$

$$\Phi_{\vec{n}}^{a, i}(x) = \left[-\binom{n_1}{n_2} \phi_{\vec{n}}^a(x) + \binom{-n_2}{n_1} H_{\vec{n}}^a(x) \right] \quad (47)$$

with the summation range³

$$n_1 \geq 1 \text{ and } |n_2| \geq 1 \text{ or } n_i = 0 \text{ and } n_j \geq 1 \quad (48)$$

The sign conventions in (47) have been chosen in accordance with (9) and (12). Symmetry breaking can be introduced by imposing further Z_2 symmetries with fixed points $y, z = \frac{1}{2}\pi R$ [9].

The interactions of the KK-modes involve KK-number conservation factors of the form (following the notation of [24])

$$\begin{aligned} \delta_{\vec{k}, \vec{l}, \vec{m}} &= \delta_{\vec{k}+\vec{l}+\vec{m}, 0} + \delta_{\vec{k}+\vec{l}-\vec{m}, 0} + \delta_{\vec{k}-\vec{l}+\vec{m}, 0} + \delta_{\vec{k}-\vec{l}-\vec{m}, 0} \\ \tilde{\delta}_{\vec{k}, \vec{l}, \vec{m}} &= -\delta_{\vec{k}+\vec{l}+\vec{m}, 0} + \delta_{\vec{k}+\vec{l}-\vec{m}, 0} - \delta_{\vec{k}-\vec{l}+\vec{m}, 0} + \delta_{\vec{k}-\vec{l}-\vec{m}, 0} \end{aligned} \quad (49)$$

The triple gauge boson interactions and the scalar coupling have the same form as in the 5-dimensional case [24]:

$$g^{\alpha\beta\gamma} = f^{abc} \left(\frac{1}{\sqrt{2}} \right)^{(1+\delta_{\vec{n}, 0}+\delta_{\vec{m}, 0}+\delta_{\vec{k}, 0})} \delta_{\vec{n}, \vec{m}, \vec{k}} \quad (50)$$

$$g_{\Phi AA}^{i\alpha\beta\gamma} = g f^{abc} \left[\frac{1}{\sqrt{2}^{\delta_{\vec{k}, 0}+1}} \tilde{\delta}_{\vec{n}, \vec{m}, \vec{k}} \frac{m_i}{R} - \frac{1}{\sqrt{2}^{\delta_{\vec{m}, 0}+1}} \tilde{\delta}_{\vec{n}, \vec{k}, \vec{m}} \frac{k_i}{R} \right] \quad (51)$$

where we have introduced the 4-dimensional coupling $g = g_6/(\sqrt{2}\pi R)$.

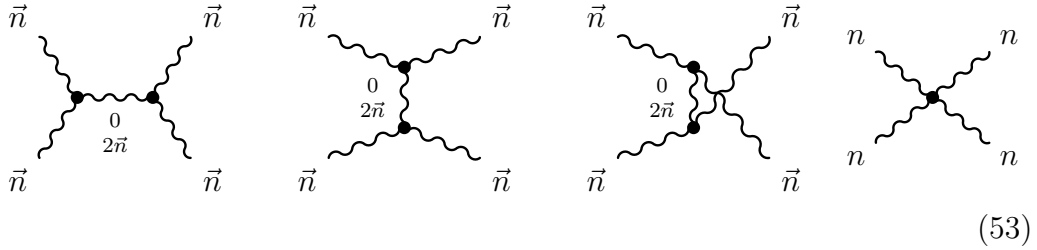
Inserting the decomposition of Φ in terms of Higgs and GBs, we find for the interaction of the Higgses

³This differs slightly from the conventions in [29] where the condition $n_1 + n_2 \geq 1$ or $n_1 = -n_2 \geq 1$ is used. Both conventions lead to the same degeneracies at each KK-level, e. g. our level $(2, \pm 1)$, $(1, \pm 2)$ corresponds to $(2, \pm 1)$, $(\pm 1, 2)$ in the conventions of [29]. For general considerations our convention seems to be more practical because we don't have to distinguish $|n_1| < |m_1|$ from $|n_1| > |n_2|$.

$$g_{HAA}^{\alpha\beta\gamma} = g f^{abc} \left[\frac{1}{\sqrt{2}^{\delta_{\vec{k},0}+1}} \tilde{\delta}_{\vec{n},\vec{m},\vec{k}} \frac{-m_1 n_2 + m_2 n_1}{m_{\vec{n}} R^2} - \frac{1}{\sqrt{2}^{\delta_{\vec{m},0}+1}} \tilde{\delta}_{\vec{n},\vec{k},\vec{m}} \frac{-k_1 n_2 + k_2 n_1}{m_{\vec{n}} R^2} \right] \quad (52)$$

4.2 Sum rules for elastic scattering

We are now ready to investigate the SRs for gauge boson scattering. We begin with the simplest case of scattering at the same KK-level where the initial and final KK-states \vec{n} are the same. Because of the KK-number conservation the contributing diagrams are



(53)

From the explicit form of the Higgs couplings (52) we see that the Higgs couplings vanish for the scattering at the same KK-level. The SR (39) obtained equivalently from the $\mathcal{M}(\mathcal{DDDW})$ WI or the cancellation of the subleading divergences in quartic gauge boson scattering then reduces to the relation discussed already in [11]

$$3m_{\vec{m}}^2 g_{\vec{n},\vec{n},\vec{m}}^{ace} g_{\vec{m},\vec{n},\vec{n}}^{ebd} = 4m_{\vec{n}}^2 g_{\vec{n},\vec{n},\vec{m}}^{ace} g_{\vec{n},\vec{n},\vec{m}}^{bde} \quad (54)$$

where we have used the Jacobi Identity. Inserting the coupling constants (50) (note that $\delta_{\vec{n},\vec{m},0} = 2\delta_{\vec{n},\vec{m}}$)

$$g_{\vec{n},\vec{n},0}^{abc} = g f^{abc}, \quad g_{\vec{n},\vec{m},\vec{n}+\vec{m}}^{abc} = \frac{g}{\sqrt{2}} f^{abc} \quad (55)$$

we find that this is indeed satisfied in the same way as in the 5-dimensional case discussed in [19].

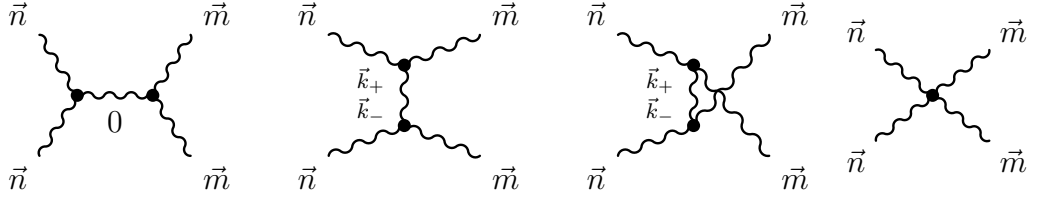
A new phenomenon in theories with more than one extra dimension is scattering at the same KK-level, but with different KK-numbers at the initial and final state, e. g. KK-exchange

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad \rightarrow \quad \vec{m} = \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} \quad (56)$$

Here, because the second component of the KK-quantum numbers can take negative values, the exchange of vector bosons with the quantum numbers

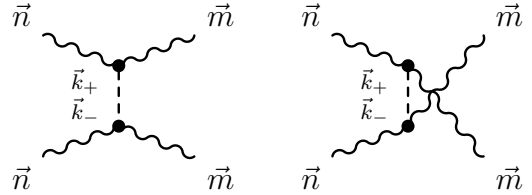
$$\vec{k}_+ = \begin{pmatrix} n_1 + n_2 \\ n_1 + n_2 \end{pmatrix} \quad \vec{k}_- = \begin{pmatrix} |n_1 - n_2| \\ -|n_1 - n_2| \end{pmatrix} \quad m_{\vec{k}_\pm} = \sqrt{2} \frac{|n_1 \pm n_2|}{R} \quad (57)$$

is possible in the t - and u -channel:



(58a)

Compared to the simpler case discussed above, the Higgs couplings (52) to gauge bosons with different KK-quantum numbers are nonvanishing, so in the t - and u -channel also the exchange of two physical scalars contributes:



(58b)

The explicit form of the Higgs couplings are given by

$$g_{HAA, \vec{k}_+, \vec{n}, \vec{m}}^{abc} = -g f^{abc} \frac{(n_1 - n_2)}{R} \quad (59)$$

$$g_{HAA, \vec{k}_-, \vec{n}, \vec{m}}^{abc} = -g f^{abc} \frac{(n_1 + n_2)}{R}$$

In contrast to the scattering processes involving the same initial and final KK-states, there is no contribution from the zero mode in the t - and u -channel. Since the cancellation of the unitarity violating terms in (54) relies on the different couplings of the zero mode and the KK-modes, this mechanism cannot be sufficient any more and the physical scalars are expected to play an important role.

To take the contributions from the additional physical scalars into account, the SR (54) has to be modified. From (39) we get (exchanging $a \leftrightarrow d$ and using that all initial and final state masses are the same):

$$\begin{aligned}
& \sum_{\vec{k}=\vec{k}_\pm} - \left(m_{\vec{k}}^2 g_{\vec{n},\vec{m},\vec{k}}^{bde} g_{\vec{n},\vec{m},\vec{k}}^{ace} + g_{HWW,\vec{n},\vec{m},\vec{k}}^{ebd} g_{HWW,\vec{n},\vec{m},\vec{k}}^{eac} \right) \\
& + \left(m_{\vec{k}}^2 g_{\vec{n},\vec{m},\vec{k}}^{bce} g_{\vec{n},\vec{m},\vec{k}}^{ade} + g_{HWW,\vec{n},\vec{m},\vec{k}}^{ebc} g_{HWW,\vec{n},\vec{m},\vec{k}}^{ead} \right) = -4m_{\vec{n}}^2 g_{\vec{n},\vec{n},0}^{abe} g_{\vec{m},\vec{m},0}^{cde} \quad (60)
\end{aligned}$$

Inserting the coupling constants of the Higgs bosons (59) and of the gauge bosons (55) this turns into

$$\begin{aligned}
& -2 \frac{g^2}{R^2} [(n_1 + n_2)^2 + (n_1 - n_2)^2] (f^{bde} f^{ace} - f^{bce} f^{ade}) \\
& = -4 \frac{g^2}{R^2} (n_1^2 + n_2^2) f^{abe} f^{cde} \quad (61)
\end{aligned}$$

This relation is guaranteed by the Jacobi Identity. Therefore the unitarity cancellations in scattering among different KK-states at the same mass-level take place because of an interplay of the KK-gauge bosons and the ‘geometric Higgs bosons’ with the peculiar form of the coupling (52).

4.3 Sum rules for inelastic scattering

We now turn to inelastic scattering at different KK-levels \vec{n} to \vec{m} . Unlike the elastic case, no general SR for inelastic scattering has been derived in [11] for a 5-dimensional gauge theory.

The contributing diagrams are the same as in (58) with the KK-momenta now given by

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad \vec{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \vec{k}_+ = \begin{pmatrix} n_1 + m_1 \\ n_2 + m_2 \end{pmatrix}, \quad \vec{k}_- = \pm \begin{pmatrix} n_1 - m_1 \\ n_2 - m_2 \end{pmatrix} \quad (62)$$

where the sign in \vec{k}_- has to be chosen so that $\pm(n_1 - m_1) \geq 0$.

From (39) we find the SR

$$\begin{aligned}
& \sum_{\vec{k}=\vec{k}_\pm} - \left(\frac{(m_{\vec{k}}^2 + m_{\vec{n}}^2 - m_{\vec{m}}^2)^2}{m_{\vec{k}}^2} g_{\vec{n},\vec{m},\vec{k}}^{bde} g_{\vec{n},\vec{m},\vec{k}}^{ace} + g_{HWW,\vec{n},\vec{m},\vec{k}}^{ebd} g_{HWW,\vec{n},\vec{m},\vec{k}}^{eac} \right) \\
& + \left(\frac{(m_{\vec{k}}^2 + m_{\vec{n}}^2 - m_{\vec{m}}^2)^2}{m_{\vec{k}}^2} g_{\vec{n},\vec{m},\vec{k}}^{bce} g_{\vec{n},\vec{m},\vec{k}}^{ade} + g_{HWW,\vec{n},\vec{m},\vec{k}}^{ebc} g_{HWW,\vec{n},\vec{m},\vec{k}}^{ead} \right) \\
& = -4m_{\vec{n}}^2 g_{\vec{n},\vec{n},0}^{abe} g_{\vec{m},\vec{m},0}^{cde} \quad (63)
\end{aligned}$$

The combinations of the masses appearing in the SR can be simplified using the relation

$$m_{\vec{k}_\pm}^2 + m_{\vec{n}}^2 - m_{\vec{m}}^2 = \frac{2}{R^2} \vec{n} \cdot (\vec{n} \pm \vec{m}) \quad (64)$$

The Higgs coupling (52) takes the form

$$g_{HAA,\vec{k}_\pm,\vec{n},\vec{m}}^{abc} = \sqrt{2}g f^{abc} \left[\frac{n_2 m_1 - n_1 m_2}{R^2 m_{\vec{k}_\pm}} \right] \quad (65)$$

Let us first discuss the simpler case $n_2 = m_2 = 0$. The Higgs couplings vanish and the situation is similar to scattering in 5 dimensions. The SR turns into

$$\begin{aligned} -2 \frac{g^2}{R^2} \left[\frac{(n^2 + nm)^2}{(m+n)^2} + \frac{(n^2 - nm)^2}{(m-n)^2} \right] (f^{bde} f^{ace} - f^{bce} f^{ade}) \\ = -4 \frac{g^2}{R^2} n^2 f^{abe} f^{cde} \end{aligned} \quad (66)$$

Again this is guaranteed by the Jacobi Identity.

The required cancellation for the general KK-momenta (62) is less obvious and requires to take the physical scalars into account. Using the expression (65) for the Higgs coupling, one can verify the relation

$$\begin{aligned} \frac{(m_{\vec{k}_\pm}^2 + m_{\vec{n}}^2 - m_{\vec{m}}^2)^2}{m_{\vec{k}_\pm}^2} g_{\vec{n},\vec{m},\vec{k}}^{bde} g_{\vec{n},\vec{m},\vec{k}}^{ace} + g_{HWW,\vec{n},\vec{m},\vec{k}_\pm}^{ebd} g_{HWW,\vec{n},\vec{m},\vec{k}_\pm}^{eac} \\ = f^{bde} f^{ace} \frac{2g^2}{R^2} \vec{n}^2 \end{aligned} \quad (67)$$

Inserting this result into the SR (63), the cancellation goes through as above.

Our discussion of gauge boson scattering shows that the physical scalar components play an important role in the cancellation, iff the initial and final state KK-momenta are not ‘parallel’, a situation that is a new phenomenon for gauge theories with more than one extra dimension.

5 Consistency of incomplete multiplets on the boundaries

An important ingredient of orbifold GUT models is the explicit symmetry breaking by matter on the orbifold fixed points, transforming under the unbroken subgroup alone. A consistency check of such a setup is provided by unitarity in the production of gauge bosons corresponding to broken generators. This has been investigated in [4] for boundary Higgs bosons in 5-dimensional SU(5) GUTs. In this example, the required cancellations occur between the KK-zero-mode and the first KK-level and rely on an apparent ‘conspiracy’ among the coupling constants. Here we provide a general

analysis for boundary fermions (other boundary fields can be treated similarly) that shows that such cancellations are ensured by the completeness relations of the KK-wavefunctions and the vanishing wavefunctions of the broken gauge bosons on the fixed points.

5.1 Matter on the boundaries

The symmetry at the orbifold fixed points can be viewed as a restricted gauge symmetry [7], since the gauge parameters associated to the broken generators vanish on the boundary. The BRST-transformations on the boundary at y_f are therefore

$$\delta_{\text{BRST}} A_\mu^a(x, y_f) = \partial_\mu c^a(x, y_f) + f^{abc} A_\mu^b(x, y_f) c^c(x, y_f) \quad (68a)$$

$$\delta_{\text{BRST}} \Phi_i^{\hat{a}}(x, y_f) = \partial_i c^{\hat{a}}(x, y_f) + f^{\hat{a}\hat{b}c} \Phi_i^{\hat{b}}(x) c^c(x, y_f) \quad (68b)$$

On the boundary, the gauge bosons only transform under the unbroken subgroup, while the higher dimensional components of the gauge bosons transform homogeneously under the unbroken subgroup and receive a ‘shift’ $\partial_i c^{\hat{a}}$ under the broken transformations. This symmetry prohibits brane-mass terms for the Φ_i in 5 dimensions and constrains the possible forms in 6 dimensions [30]. For symmetry breaking with mixed BCs, the structure of the transformations on the boundary is more complicated because the broken gauge (and ghost) fields are also nonvanishing at the boundary and the separation of broken and unbroken gauge transformations is lost.

On the boundaries we can add (possibly chiral) matter transforming under the unbroken subgroup with generators $\tau_{L/R}$ satisfying the Lie algebra (35a). The lagrangian for fermions on the boundary is

$$\begin{aligned} \mathcal{L}_f &= i\bar{\psi}_i \not{\partial} \psi + \bar{\psi}_i A_a(y_f) (\tau_{Lij}^a (\frac{1-\gamma^5}{2}) + \tau_{Rij}^a (\frac{1+\gamma^5}{2})) \psi_j \\ &= i\bar{\psi}_i \not{\partial} \psi + \bar{\psi}_i A_\alpha (\mathcal{T}_{Lij}^\alpha (\frac{1-\gamma^5}{2}) + \mathcal{T}_{Rij}^\alpha (\frac{1+\gamma^5}{2})) \psi_j \end{aligned} \quad (69)$$

with

$$\mathcal{T}_{L/R}^\alpha = \tau_{L/R}^a f^\alpha(y_f) \quad (70)$$

It is not possible to add Yukawa-Interactions of brane fermions to the scalar components of the bulk gauge bosons, since this violates the shift-symmetry $\propto \partial_i c^{\hat{a}}$ from (68). To use a component of the higher dimensional gauge fields as Higgs boson, one either has to put the fermions in the bulk or introduce a non-local coupling [20] that can be generated from mixing with bulk fermions.

The lagrangian of the KK-modes is invariant under the BRST transformations (21) and

$$\delta_{\text{BRST}} \psi_{L/Ri} = i c^\alpha(x) \mathcal{T}_{L/Rij}^\alpha \psi_{L/Rj} \quad (71)$$

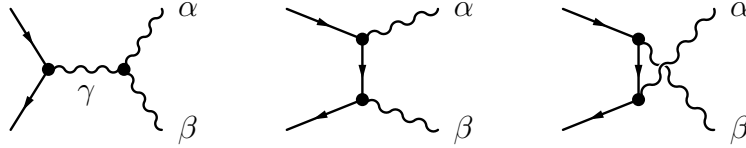
iff the relation

$$[\mathcal{T}_{L/R}^\alpha, \mathcal{T}_{L/R}^\beta] = g^{\alpha\beta\gamma} \mathcal{T}_{L/R}^\gamma \quad (72)$$

is satisfied, which also ensures the nilpotency of the BRST transformation. We will discuss this equation below, where the same relation appears as condition for unitarity. It should be noted that on the right hand side also the coupling constants $g^{\hat{\alpha}\hat{\beta}\gamma}$ with broken indices can appear, in contrast to the BRST transformations restricted to the brane (68). In the KK-picture the broken indices appear because the KK-modes are not localized in the extra dimension and the reduced symmetry is not apparent.

5.2 Unitarity in gauge boson production

We turn to the SRs for scattering of brane fermions into gauge bosons. For the scattering into unbroken gauge bosons, the KK-excitations can couple to the brane fermions and to an intermediate unbroken gauge boson, so that the contributing diagrams are

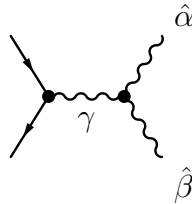

(73)

The SR obtained from the cancellation of the leading divergences (or equivalently the WI with a single contraction) is just the Lie algebra (72). Inserting the definitions (11a) and (70), we find that this relation is satisfied because of the completeness of the KK-wavefunctions and the Lie algebra (35a). Explicitly, the RHS becomes

$$\begin{aligned} g^{\alpha\beta\gamma} \mathcal{T}_{L/R}^\gamma &= f^{abc} \tau_{L/R}^c \sum_\gamma f^\gamma(y_f) \int d^N y f^\alpha(y) f^\beta(y) f^\gamma(y) \\ &= f^{abc} \tau_{L/R}^c f^\alpha(y_f) f^\beta(y_f) \end{aligned} \quad (74)$$

Thus the same KK-wavefunctions appear as on the left hand side and (72) is reduced to the simple Lie algebra of the $\tau_{L/R}^a$.

Only the s -channel diagrams contribute to the production of broken gauge bosons, since the boundary fermions completely decouple from the broken gauge bosons:


(75)

Therefore the SR (72) reduces to

$$g^{\hat{\alpha}\hat{\beta}\gamma}\mathcal{T}_{L/R}^\gamma = 0. \quad (76)$$

Exploiting the completeness relation of the KK-wavefunctions once more, we see that this relation is indeed satisfied because the broken wavefunctions vanish on the boundary:

$$\begin{aligned} g^{\hat{\alpha}\hat{\beta}\gamma}\mathcal{T}_{L/R}^\gamma &= f^{\hat{a}\hat{b}c}\tau_{L/R}^c \sum_\gamma f^\gamma(y_f) \int d^N y f^{\hat{\alpha}}(y) f^{\hat{\beta}}(y) f^\gamma(y) \\ &= f^{\hat{a}\hat{b}c}\tau_{L/R}^c f^{\hat{\alpha}}(y_f) f^{\hat{\beta}}(y_f) = 0 \end{aligned} \quad (77)$$

For massive chiral fermions, subleading divergences appear whose cancellation requires the SR (104). Masses of the chiral brane-fermions are not included in our present setting and have to be generated by a further breaking of the unbroken subgroup, e. g. by a Higgs boson localized on the brane. The investigation of such constructions is beyond the scope of the present work.

6 Summary and outlook

We have performed the KK-decomposition of a general gauge theory on an C/Z_2^n orbifold and determined consistent boundary conditions that allow BRST quantization and to derive WIs. This yields a new demonstration of the consistency of orbifold symmetry breaking and of the Dirichlet boundary conditions currently employed for models of EWSB without Higgs bosons [11, 12]. On the other hand, mixed boundary conditions turn out to be inconsistent without the introduction of a Higgs multiplet on the boundary. In the presence of such a brane Higgs multiplet, the GBs are a mixture of the unphysical components of the boundary Higgs with the higher dimensional components of the gauge bosons [24]. Dirichlet BCs arise from the limit of an infinite vacuum expectation value of a boundary Higgs. As an extension of our work, the consistency of such a setup could be further clarified by quantizing such theories including the boundary Higgs, taking the limit $v \rightarrow \infty$ at the end.

Recently, a Higgsless mechanism has also been proposed for the generation of fermion masses by BCs [13]. The application of the approach presented here to this construction is given elsewhere [31].

In section 3 we have shown that on tree level the structure of the lagrangian of a SBGT is fixed by imposing a finite set of WIs for 4-point functions without external goldstone bosons. The conditions derived from the

WIs include the unitarity-SRs derived from tree level unitarity in [16, 17, 18]. The same SRs are also valid in compactified higher dimensional gauge theories broken by orbifold BCs, provided the sum over the KK-tower converges. The introduction of a cutoff or regularization of the KK-sums can upset gauge invariance and requires further considerations.

In section 4 the SRs have been applied to a 6-dimensional gauge theory. We have shown that the physical scalar components of the gauge bosons play an important role in ensuring the unitarity cancellations in gauge boson scattering where the final state KK-momenta are not parallel to that of the initial state gauge bosons. The clarification of this mechanism should prove useful in extending the discussion of partial wave unitarity in KK-theories [14, 15] to 6-dimensional models. In section 5 we have demonstrated the consistency of placing reduced multiplets at the orbifold fixed points.

After this work was completed, a discussion of unitarity in the Higgsless models of [12] appeared [32], where it is pointed out that partial wave unitarity might be violated at a scale of $\sqrt{s} \sim 2 \text{ TeV}$ despite the fulfillment of the unitarity sum rules if the mass of the first KK-excitation is too large. This poses a problem for the warped version of the model while in a flat space model no such problems have been found. While the present work was concerned with gauge theories on separable background metrics, our approach will also be useful for studying unitarity of gauge boson scattering in a warped background and for answering open questions.

Acknowledgments

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A Kaluza-Klein lagrangian and couplings

The $(4 + N)$ -dimensional Yang Mills lagrangian is

$$\mathcal{L}_{4+N} = -\frac{1}{4}F_{AB}^a(x, y)F^{a,AB}(x, y) \quad (78)$$

with the field strength

$$F_{AB}^a(x, y) = \partial_A A_B^a(x, y) - \partial_B A_A^a(x, y) + f^{abc} A_A^b(x, y) A_B^c(x, y) \quad (79)$$

Here we include the higher dimensional gauge coupling g_D in the structure constants. The BRST transformations are

$$\begin{aligned}
\delta_{\text{BRST}} A_A^a(x, y) &= \partial_A c^a(x, y) + f^{abc} A_A^b(x, y) c^c(x, y) \\
\delta_{\text{BRST}} c^a(x, y) &= -\frac{1}{2} f^{abc} c^b(x, y) c^c(x, y) \\
\delta_{\text{BRST}} \bar{c}^a(x, y) &= B^a(x, y) \\
\delta_{\text{BRST}} B^a(x, y) &= 0
\end{aligned} \tag{80}$$

where the equation of motion of the auxiliary field B^a is

$$B^a = -\frac{1}{\xi} G^a = -\frac{1}{\xi} (\partial_\mu A^{a,\mu}(x, y) - \xi \partial_i \Phi^{a,i}(x, y)) \tag{81}$$

The complete lagrangian of the KK-modes is

$$\begin{aligned}
\mathcal{L}_{KK} &= -\frac{1}{4} (\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha) (\partial^\mu A^{\alpha,\nu} - \partial^\nu A^{\alpha,\mu}) + \frac{1}{2} m_\alpha^2 A_\mu^\alpha A^{\alpha,\mu} \\
&\quad - m_{\alpha_i} \partial_\mu \Phi^{\alpha,i} A^{\alpha,\mu} - g^{\alpha\beta\gamma} \partial_\mu A_\nu^\alpha A^{\beta,\mu} A^{\gamma,\nu} - \frac{1}{4} g^{\alpha\beta\gamma\delta} A_\mu^\alpha A_\nu^\beta A^{\gamma,\mu} A^{\delta,\nu} \\
&\quad + \frac{1}{2} \partial_\mu \Phi_i^\alpha \partial^\mu \Phi^{\alpha,i} - \frac{1}{2} [m_\alpha^2 \Phi_i^\alpha \Phi^{\alpha,i} - (m_{\alpha_i} \Phi^{\alpha,i})^2] \\
&\quad - \frac{1}{2} T_{\beta\gamma}^\alpha A^{\alpha,\mu} \Phi_i^\beta \overleftrightarrow{\partial}_\mu \Phi^{\gamma,i} + \frac{1}{2} g_{\Phi AA}^{i\alpha\beta\gamma} \Phi^{\alpha,i} A_\mu^\beta A^{\mu,\gamma} + \frac{1}{4} g_{A^2\Phi^2}^{\alpha\beta\gamma\delta} A_\mu^\alpha A_\nu^\beta \Phi^{\gamma,i} \Phi_i^\delta \\
&\quad - \frac{1}{2} T_{\beta\gamma}^\alpha (m_{\alpha_j} \Phi_i^\alpha - m_{\alpha_i} \Phi_j^\alpha) \Phi^{\beta,i} \Phi^{\gamma,j} - \frac{1}{4} g_{\Phi^4}^{\alpha\beta\gamma\delta} \Phi_i^\alpha \Phi_j^\beta \Phi^{\gamma,i} \Phi^{\delta,j}
\end{aligned} \tag{82}$$

with the coupling constants defined in (11) and

$$g^{\alpha\beta\gamma\delta} = f^{abe} f^{cde} \int d^N y f^\alpha(y) f^\beta(y) f^\gamma(y) f^\delta(y) \tag{83a}$$

$$g_{\Phi AA}^{i\alpha\beta\gamma} = m_{\beta_i} T_{\beta\alpha}^\gamma + m_{\gamma_i} T_{\gamma\alpha}^\beta \tag{83b}$$

$$g_{A^2\Phi^2}^{\alpha\beta\gamma\delta} = 2 f^{abe} f^{cde} \int d^N y f^\alpha(y) f^\beta(y) g^\gamma(y) g^\delta(y) \tag{83c}$$

$$g_{\Phi^4}^{\alpha\beta\gamma\delta} = f^{abe} f^{cde} \int d^N y g^\alpha(y) g^\beta(y) g^\gamma(y) g^\delta(y) \tag{83d}$$

B Parameterization of the general lagrangian

We give our parameterization of the general lagrangian with the particle spectrum of a SGBT used in the calculation of the WIs. Apart from terms

$\propto \epsilon^{\mu\nu\rho\sigma} W_\mu W_\nu W_\rho W_\sigma$, the most general renormalizable interaction lagrangian for these fields is

$$\begin{aligned}
\mathcal{L}_{\text{int}} = & -f^{abc} W_{b,\mu} W_{c,\nu} \partial^\mu W_a^\nu - \frac{1}{4} g_{W^4}^{abcd} W_{a,\mu} W_{b,\nu} W_c^\mu W_d^\nu - \frac{1}{2} t_{ab}^c (\phi_a \overleftrightarrow{\partial}_\mu \phi_b) W_c^\mu \\
& + \frac{g_{\phi WW}^{abc}}{2} \phi_a W_b^\mu W_{c,\mu} - \frac{1}{2} T_{ij}^a (H_i \overleftrightarrow{\partial}_\mu H_j) W_a^\mu + g_{H\phi W}^{iab} (\phi_a \overleftrightarrow{\partial}_\mu H_i) W_b^\mu \\
& + \frac{1}{2} g_{HWW}^{iab} H_i W_{a,\mu} W_b^\mu + \frac{1}{4} g_{\phi^2 W^2}^{abcd} \phi_a \phi_b W_{c,\mu} W_d^\mu + \frac{1}{4} g_{H^2 W^2}^{abij} H_i H_j W_a^\mu W_{b,\mu} \\
& + \frac{1}{2} g_{H\phi W^2}^{abci} H_i \phi_a W_{b,\mu} W_c^\mu + \frac{1}{2} g_{\phi^2 H}^{abi} \phi_a \phi_b H_i + \frac{1}{2} g_{\phi H^2}^{aij} \phi_a H_i H_j \\
& + \frac{1}{3!} g_{\phi^3}^{abc} \phi_a \phi_b \phi_c + \frac{1}{3!} g_{H^3}^{ijk} H_i H_j H_k + \text{quartic scalar interactions}
\end{aligned} \tag{84a}$$

The lagrangian of the fermions is parametrized by

$$\begin{aligned}
\mathcal{L}_f = & i\bar{\psi}_i \not{\partial} \psi + \bar{\psi}_i W_a (\tau_{Lij}^a (\frac{1-\gamma^5}{2}) + \tau_{Rij}^a (\frac{1+\gamma^5}{2})) \psi_j \\
& + \bar{\psi}_i \phi_a (g_{\phi ij}^a (\frac{1-\gamma^5}{2}) + g_{\phi ij}^{a\dagger} (\frac{1+\gamma^5}{2})) \psi_j + \bar{\psi}_i H_k (g_{Hij}^k (\frac{1-\gamma^5}{2}) + g_{Hij}^{k\dagger} (\frac{1+\gamma^5}{2})) \psi_j
\end{aligned} \tag{84b}$$

This can be compared with the lagrangian of a SGBT

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu} + \frac{1}{2} D_\mu \phi_A D^\mu \phi_A - V(\phi) \\
& + i\bar{\psi}_i \not{\partial} \psi_i + \bar{\psi}_i \phi_A (X_{ij}^A (\frac{1-\gamma^5}{2}) + X_{ij}^{A\dagger} (\frac{1+\gamma^5}{2})) \psi_j
\end{aligned} \tag{85}$$

with

$$F_{a\mu\nu} = \partial_\mu A_{a,\nu} - \partial_\nu A_{a,\mu} + f^{abc} A_{b,\nu} A_{c,\mu} \tag{86}$$

$$D_\mu \phi_A = \partial_\mu \phi_A + T_{AB}^a W_{a,\mu} \phi_B \tag{87}$$

$$V(\phi) = \frac{g_2^{AB}}{2} \phi^A \phi^B + \frac{g_3^{ABC}}{3!} \phi^A \phi^B \phi^C + \frac{g_4^{ABCD}}{4!} \phi^A \phi^B \phi^C \phi^D \tag{88}$$

$$D_\mu \psi_i = \partial_\mu \psi_i - iW_{a,\mu} (\tau_{Lij}^a (\frac{1-\gamma^5}{2}) + \tau_{Rij}^a (\frac{1+\gamma^5}{2})) \psi_j \tag{89}$$

Inserting the parameterization (34) for the generators in the representation of the scalars, we find that the definitions of T_{ij}^a , t_{ab}^c and $g_{H\phi W}$ agree with those in the lagrangian (84a). The 2 scalar-2 gauge boson coupling is given by the anticommutator of representation matrices:

$$g_{\phi^2 W^2}^{ABcd} = -\{T^c, T^d\}_{AB} \tag{90}$$

The cubic scalar gauge boson couplings originate from the contraction of the anticommutator $\{T^a, T^b\}$ with a vacuum expectation value ϕ_0 :

$$g_{\phi WW}^{Abc} = g_{\phi^2 W^2}^{ABbc} \phi_{0B} \tag{91}$$

The triple scalar couplings can be expressed through the terms in the scalar potential by

$$g_{\Phi^3}^{ABC} = -g_3^{ABC} - g_4^{iABC} v_i \quad (92)$$

C Form of the Goldstone boson couplings

The relations (37) follow from the condition that the gauge transformation of the vacuum expectation value must be in the Goldstone boson direction

$$T_{bc}^a \phi_{0c} = \begin{pmatrix} m_{W_a} \\ 0 \end{pmatrix} \delta_{a,b} \quad (93)$$

Using (91) this implies

$$g_{\phi WW}^{Abc} = (m_b T_{bA}^c + m_c T_{cA}^b) \quad (94)$$

Acting on the vacuum expectation value ϕ_0 with a commutator of two generators in the representation of the scalars results in:

$$([\mathbf{T}^a, \mathbf{T}^b]) \phi_0 = f^{abc} \begin{pmatrix} m_{W_c} \\ 0 \end{pmatrix} \Rightarrow \begin{cases} t_{cb}^a m_{W_b} - t_{ca}^b m_{W_a} = f^{abc} m_{W_c} \\ m_a g_{H\phi W}^{iab} - m_b g_{H\phi W}^{iba} = 0 \end{cases} \quad (95)$$

Together with (94), the first relation implies the relation (37a) [22] while the second one implies (37b). Similarly, contracting the transformation law of the Yukawa couplings (35c) with the vacuum expectation value ϕ_{0A} and using the condition that the fermions get their masses from the coupling to the scalars

$$X_{ij}^A \phi_{0A} = X_{ij}^{A\dagger} \phi_{0A} = -\delta_{ij} m_i \quad (96)$$

one can derive the fermion-Goldstone boson coupling (37c).

The remaining relations on the cubic GB couplings (99c), (100a) and (100c) are consequences of the invariance of the scalar potential

$$\frac{\partial V(\phi)}{\partial \phi_A} T_{AB}^a \phi_B = 0 \quad (97)$$

as can be seen by taking two derivatives with respect to ϕ and setting $\phi = \phi_0$.

D Results from the WIs

In this appendix we collect the results of the WIs for the 3- and 4-point functions with up to 4 contractions. In the calculations the general lagrangian (84) has been used. No symmetry relations among the coupling

constants have been assumed, but only results of previously evaluated WIs have been used to simplify the calculations. The SRs obtained in this way allow to express all coupling constants in terms of the input parameters

$$f^{abc}, \tau_{L/Rij}^a, g_{Hij}^h, g_{HWW}^{iab}, T_{ij}^a, g_{H^3}^{ijk}, g_{H^4}^{ijkl} \quad (98)$$

This is not a *minimal* set of input parameters since they are subject to constraints arising from the Lie algebra, the Jacobi Identities and other symmetry relations. The detailed calculations of the WIs can be found in [22].

D.1 Cubic Goldstone boson couplings

The couplings of one Goldstone boson to two physical particles are determined by the WIs with one contraction.

$$\begin{array}{c} \text{Diagram: A double line (Goldstone boson) connects a shaded vertex to a wavy line (physical particle). The vertex also has a dashed line entering from above.} \\ = 0 \end{array} \Rightarrow g_{H\phi W}^{iab} = -\frac{1}{2m_{W_a}} g_{HWW}^{iab} \quad (99a)$$

$$\begin{array}{c} \text{Diagram: A double line (Goldstone boson) connects a shaded vertex to a horizontal arrow (physical particle). The vertex also has a dashed line entering from above.} \\ = 0 \end{array} \Rightarrow g_{\phi ij}^a = -\frac{i}{m_{W_a}} (m_{f_i} \tau_{Lij}^a - m_{f_j} \tau_{Rij}^a) \quad (99b)$$

$$\begin{array}{c} \text{Diagram: A double line (Goldstone boson) connects a shaded vertex to a dashed line (physical particle). The vertex also has a dashed line entering from above.} \\ = 0 \end{array} \Rightarrow g_{\phi H^2}^{aij} = \frac{1}{m_{W_a}} T_{ij}^a (m_i^2 - m_j^2) \quad (99c)$$

$$\begin{array}{c} \text{Diagram: A double line (Goldstone boson) connects a shaded vertex to a wavy line (physical particle). The vertex also has a wavy line entering from above.} \\ = 0 \end{array} \Rightarrow g_{\phi WW}^{abc} = \frac{1}{m_{W_a}} f^{abc} (m_{W_b}^2 - m_{W_c}^2) \quad (99d)$$

(in all diagrams, the insertion of the operator $(\partial_\mu W^\mu - m_W \phi)$ is represented by a double line). To obtain the relations for the couplings of 2 Goldstone bosons to one physical particle, one has to consider the WIs (32) with two contractions

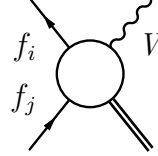
$$\begin{array}{c} \text{Diagram: A double line (Goldstone boson) connects a shaded vertex to a double line (Goldstone boson). The vertex also has a dashed line entering from above.} \\ = 0 \end{array} \Rightarrow g_{\phi^2 H}^{abi} = -\frac{m_{H_i}^2}{2m_{W_a} m_{W_b}} g_{HWW}^{iab} \quad (100a)$$

$$\begin{array}{c} \text{Diagram: A double line (Goldstone boson) connects a shaded vertex to a double line (Goldstone boson). The vertex also has a wavy line entering from above.} \\ = 0 \end{array} \Rightarrow m_{W_a} m_{W_c} t_{ac}^b = \frac{1}{2} f^{bac} (m_{W_b}^2 - m_{W_a}^2 - m_{W_c}^2) \quad (100b)$$

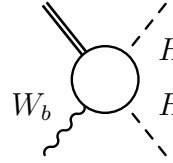
$$\begin{array}{c} \text{Diagram: A double line (Goldstone boson) connects a shaded vertex to a double line (Goldstone boson). The vertex also has a double line entering from above.} \\ = 0 \end{array} \Rightarrow g_{\phi^3}^{abc} = 0 \quad (100c)$$

D.2 Gauge couplings of physical particles

The Lie algebra structure of the couplings of the physical particles, i.e. the Higgs bosons, gauge bosons and fermions, arises from WIs for 4-point functions with one contraction, together with the quartic gauge couplings and the $2W2H$ coupling:



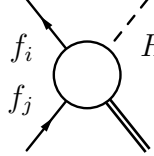
$$\Rightarrow \begin{cases} [\tau_L^a, \tau_L^b]_{ij} - i f^{abc} \tau_{Lij}^c = 0 \\ [\tau_R^a, \tau_R^b]_{ij} - i f^{abc} \tau_{Rij}^c = 0 \end{cases} \quad (101)$$



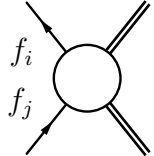
$$\Rightarrow \begin{cases} [T^a, T^b]_{ij} - [\frac{g_{HWW}^a}{2m_{W_c}}, \frac{g_{HWW}^b}{2m_{W_c}}]_{ij} = f^{abc} T_{ij}^c \\ \{T_{ik}^a, T_{kj}^b\} - \{\frac{g_{HWW}^a}{2m_{W_c}}, \frac{g_{HWW}^b}{2m_{W_c}}\}_{ij} = -g_{H^2 W^2}^{abij} \end{cases} \quad (102)$$

D.2.1 Yukawa couplings

The symmetry conditions of the Yukawa couplings (35c) can be obtained from the WI for 2 fermions, one gauge boson and one Higgs boson and the WI for two fermions and 2 contractions:



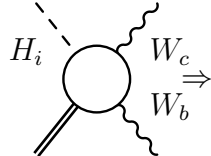
$$\Rightarrow 0 = -i \frac{g_{HWW}^{hba}}{2m_{W_a}} g_{\phi ij}^a - i(g_{Hij}^h T_{hk}^b) - g_{Hil}^k \tau_{Llj}^b + \tau_{Ril}^b g_{Hlj}^k \quad (103)$$



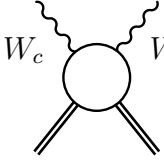
$$\Rightarrow i g_{\phi il}^b \tau_{Llj}^a - i \tau_{Ril}^a g_{\phi lj}^b = g_{\phi ij}^c t_{cb}^a - g_{Hij}^k \frac{g_{HWW}^{kab}}{2m_{W_b}} \quad (104)$$

D.3 Goldstone-gauge boson couplings

The components of the graded Jacobi Identity (40) are reproduced by the the WI for the $3WH$ amplitude with one contraction and the $4W$ WI with 2 contractions:

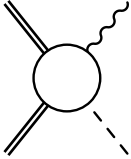


$$\Rightarrow m_{W_a} g_{H\phi W^2}^{abc i} = -g_{HWW}^{icd} f^{abd} - g_{HWW}^{ibd} f^{acd} - \frac{g_{HWW}^{iad}}{2m_{W_d}} g_{\phi WW}^{dbc} + T_{ij}^a g_{HWW}^{jbc} \quad (105)$$



$$\Rightarrow m_{W_a} m_{W_b} g_{\phi^2 W^2}^{abcd} - \frac{1}{2} g_{HWW}^{iab} g_{HWW}^{icd} = m_{W_b} g_{\phi WW}^{ecd} t_{be}^a + f^{ace} f^{dbe} (m_{W_d}^2 - m_{W_e}^2) - f^{cbe} f^{dae} (m_{W_c}^2 - m_{W_e}^2) \quad (106)$$

The ia component of the Lie algebra (35b) follows from the $3WH$ WI with two contractions:

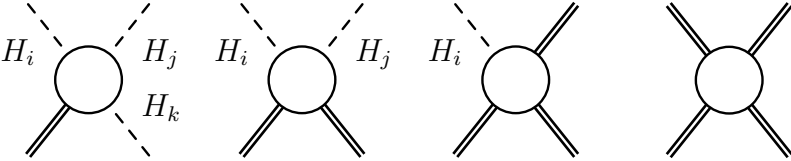


$$\Rightarrow -\frac{1}{2m_{W_e}^2} g_{HWW}^{ice} f^{bae} (m_{W_b}^2 - m_{W_a}^2 - m_{W_e}^2) + \frac{1}{2m_{W_e}^2} g_{HWW}^{ieb} f^{cae} (m_{W_c}^2 - m_{W_a}^2 - m_{W_e}^2) - g_{HWW}^{jab} T_{ji}^c + T_{ji}^b g_{HWW}^{jac} = -f^{bce} g_{HWW}^{iae} \quad (107)$$

The ab component results from the the $4W$ WI with 3 contractions given in the main text in (39).

D.4 Scalar potential

The components of the invariance condition (42) of the scalar potential are obtained by the WIs with external Higgs bosons and the WI with 4 contractions:



$$\quad (108)$$

Since the components of (42) written in terms of the input parameters (98) are somewhat involved and not needed in this work we refer the reader to [22] for the explicit expressions.

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